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CR-Warped Product Submanifolds of Lorentzian Manifolds*

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Abstract. In this paper, we study warped product CR-submanifolds of a Lorentzian Sasakian manifold. We show that the warped product of the type $M = N_\perp \times_f N_T$ in a Lorentzian Sasakian manifold is simply CR-product and obtain a characterization of CR-warped product submanifolds.

1. Introduction

Warped product manifolds were introduced by Bishop and O'Neill in [3] to construct new examples of negatively curved manifolds. These manifolds are obtained by warping the product metric of a product manifold onto the fibers and thus provide a natural generalization to the product manifolds. Let (N_1, g_1) and (N_2, g_2) be semi-Riemannian manifolds of dimensions m and n, respectively and f, a positive differentiable function on N_1. Then the warped product [3] of (N_1, g_1) and (N_2, g_2) with warping function f is defined to be the product manifold $M = N_1 \times N_2$ with metric tensor $g = g_1 + f^2 g_2$. The warped product manifold $(N_1 \times N_2, g)$ is denoted by $N_1 \times_f N_2$. If U is tangent to $M = N_1 \times_f N_2$ at (p, q) then

$$\|U\|^2 = \|d\pi_1 U\|^2 + f^2(p)\|d\pi_2 U\|^2,$$

where π_1 and π_2 are the canonical projections of M onto N_1 and N_2, respectively. The function f is called the *warping function* of the warped product manifold. In particular, if the warping function is constant, then the warped product manifold M is said to be *trivial*. Let X be vector field on N_1 and Z be vector field on N_2, then from Lemma 7.3 of [3], we have

$$\nabla_X Z = \nabla_Z X = \left(\frac{Xf}{f}\right) Z, \tag{1.1}$$

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where ∇ is the Levi-Civita connection on M. Let $M = N_1 \times_f N_2$ be a warped product manifold, this means that N_1 is totally geodesic and N_2 is totally umbilical submanifold of M, respectively.

The notion of CR-submanifolds of Kaehler manifolds was introduced by A. Bejancu [2] as a generalization of totally real and holomorphic submanifolds of a Kaehler manifold. Later, the concept of CR-submanifold has been also considered in various manifolds. In [6] and [1], as analogous of submanifolds of Lorentzian paracontact and Lorentzian manifolds, respectively. Furthermore H. Gill and K.K. Dube have recently introduced generalized CR-submanifolds of a trans Lorentzian Sasakian manifold [7].

Recently, B.Y. Chen has introduced the notion of CR-warped product in Kaehler manifolds and showed that there exist no proper warped product CR-submanifolds in the form $M = N_\perp \times {}_f N_T$ in a Kaehler manifold. He considered only the warped product of the type $M = N_T \times {}_f N_\perp$ and called it a CR-warped product submanifold [4, 5]. Later on, Hasegawa and Mihai proved that warped product CR-submanifolds $N_\perp \times {}_f N_T$ in Sasakian manifolds are trivial i.e. simply contact CR-product submanifolds, where N_T and $N_\perp$ are $\phi-$invariant and anti-invariant submanifolds of a Sasakian manifold respectively [8].

In this paper, we study warped product CR-submanifolds of a Lorentzian Sasakian manifold. We, show that the warped product in the form $M = N_\perp \times {}_f N_T$ does not exist except for the trivial case, where N_T and $N_\perp$ are invariant and anti-invariant submanifolds of a Lorentzian Sasakian manifold $\bar{M}$, respectively. Also, we obtain a characterization result of the warped product CR-submanifold of the type $M = N_T \times {}_f N_\perp$.

2. Preliminaries

A $(2m+1)-$dimensional manifold $\bar{M}$ is said to be a *Lorentzian almost contact* manifold with an almost contact structure and compatible Lorentzian metric, $(\bar{M},\ \phi, \xi, \eta, g)$, that is, ϕ is a $(1,1)$ tensor field, ξ is a structure vector field, η is a $1-$form and g is Lorentzian metric on $\bar{M}$, satisfying [1]:

$$\phi^2 = -X + \eta(X)\xi, \eta(\xi) = 1, \phi\xi = 0, \eta \circ \phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y),\ \eta(X) = -g(X,\xi) \tag{2.2}$$

for all $X, Y \in T\bar{M}$. It is *Lorentzian Sasakian* if

$$\begin{cases} (\bar{\nabla}_X \phi)Y = -g(X,Y)\xi - \eta(Y)X, \\ \quad \bar{\nabla}_X \xi = -\phi X, \end{cases} \tag{2.3}$$

for any vector fields X, Y on $\bar{M}$, where $\bar{\nabla}$ denotes the Levi-Civita connection with respect to g.

Let M be a $n-$dimensional submanifold of a Lorentzian almost contact manifold $\bar{M}$ with Lorentzian almost contact structure (ϕ, ξ, η, g). Let the induced connection on M be denoted by ∇. Then the Gauss and Weingarten Formulae are respectively given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.4}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.5}$$

for any $X,\ Y \in TM$ and $N \in T^{\perp}M$, where TM is the Lie algebra of vector fields in M and $T^{\perp}M$ is the set of all vector fields normal to M. $\nabla^{\perp}$ is the connection in the normal bundle, h the second fundamental form and A_N is the Weingarten endomorphism associated with N. It is easy to see that

$$g(A_N X, Y) = g(h(X, Y), N). \tag{2.6}$$

For any $X \in TM$, we write

$$\phi X = PX + FX, \tag{2.7}$$

where PX is the tangential component and FX is the normal component of ϕX. Similarly for $N \in T^{\perp}M$, we write

$$\phi N = tN + fN, \tag{2.8}$$

where tN is the tangential component and fN is the normal component of ϕN.

The covariant derivatives of the tensor fields ϕ, P and F are defined as

$$(\bar{\nabla}_X \phi)Y = \bar{\nabla}_X \phi Y - \phi \bar{\nabla}_X Y, \forall\ X, Y \in T\bar{M} \tag{2.9}$$

$$(\bar{\nabla}_X P)Y = \nabla_X PY - P\nabla_X Y,\ \forall\ X, Y \in TM \tag{2.10}$$

$$(\bar{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y, \forall\ X, Y \in TM. \tag{2.11}$$

Moreover, for a Lorentzian Sasakian manifold we have

$$(\bar{\nabla}_X P)Y = A_{FY}X + th(X, Y) - g(X, Y)\xi - \eta(Y)X, \tag{2.12}$$

$$(\bar{\nabla}_X F)Y = fh(X, Y) - h(X, PY). \tag{2.13}$$

A submanifold M of a Lorentzian almost contact manifold, $(\bar{M}^{2m+1}, \phi, \eta, \xi, g)$ is called *CR-submanifold* if it admits an invariant distribution $\mathcal{D}$ whose orthogonal complementary distribution $\mathcal{D}^{\perp}$ is anti-invariant i.e., $TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \langle \xi \rangle$ with $\phi(\mathcal{D}_x) \subseteq \mathcal{D}_x$ and $\phi(\mathcal{D}^{\perp}_x) \subset T^{\perp}_x M$, for every $x \in M$.

Note that ξ is a timelike vector field and all vector field in $\mathcal{D} \oplus \mathcal{D}^{\perp}$ are space like. Denoting orthogonal complementary subbundle to $\phi\mathcal{D}^{\perp}$ in $T^{\perp}M$ by μ, then we have

$$T^{\perp}M = \phi\mathcal{D}^{\perp} \oplus \mu.$$

Invariant and anti-invariant submanifolds are the special cases of CR-submanifolds. A submanifold M called an *invariant* submanifold if $\mathcal{D}^{\perp} = \{0\}$ and M is said to be an *anti-invariant* submanifold if $\mathcal{D} = \{0\}$. A CR-submanifold is *proper* if neither $\mathcal{D} = \{0\}$ nor $\mathcal{D}^{\perp} = \{0\}$.

In the following section we shall investigate the warped products of the type $M = N_T \times {}_f N_{\perp}$ and $M = N_{\perp} \times {}_f N_T$, where N_T and $N_{\perp}$ are invariant and anti-invariant submanifolds of a Lorentzian Sasakian manifold $\bar{M}$. A warped product CR-submanifold is simply *CR-product* with the integrable distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ if the warping function f is constant.

3. Warped product CR-submanifolds

Throughout the section structure vector field ξ is either tangent to the invariant submanifold N_T or tangent to the anti-invariant submanifold $N_{\perp}$. There are two types of warped product CR-submanifolds of a Lorentzian Sasakian manifold $\bar{M}$, namely $N_{\perp} \times {}_f N_T$ and $N_T \times {}_f N_{\perp}$. In the following theorem we deal the warped product CR-submanifold of the type $N_{\perp} \times {}_f N_T$.

Theorem 3.1. *Let $M = N_{\perp} \times {}_f N_T$ be a warped product CR-submanifold of a Lorentzian Sasakian manifold $\bar{M}$, where N_T and $N_{\perp}$ are invariant and anti-invariant submanifolds of $\bar{M}$, respectively. Then M is CR-product.*

Proof. For any $X \in TN_T$ and $Z \in TN_{\perp}$, by (1.1) we deduced that

$$\nabla_X Z = \nabla_Z X = (Z \ln f)X. \tag{3.1}$$

There are two cases arise:

(1) When $\xi \in TN_T$, then $\bar{\nabla}_Z \xi = -\phi Z$, i.e., $h(Z, \xi) = -\phi Z$ and $\nabla_Z \xi = 0$. On using (3.1) we get

$$(Z \ln f)\xi = 0, \ \forall \ Z \in TN_{\perp}. \tag{3.2}$$

(2) When $\xi \in TN_{\perp}$, then for any $X \in TN_T$ we have $\bar{\nabla}_X \xi = -\phi X = -PX$. This means that $h(X, \xi) = 0$ and $\nabla_X \xi = -\phi X$. Using (3.1) we get

$$(\xi \ln f)X = -\phi X, \ \forall \ X \in TN_T. \tag{3.3}$$

Taking product in (3.3) with $X \in TN_T$ thus, we obtain

$$(\xi \ln f)\|X\|^2 = 0, \ \forall \ X \in TN_T. \tag{3.4}$$

Now for any $X \in TN_T$ and $Z \in TN_{\perp}$, we have

$$\begin{aligned} g(h(X, \phi X), \phi Z) &= g(\bar{\nabla}_X \phi X, \ \phi Z) \\ &= g(\phi \bar{\nabla}_X X + (\bar{\nabla}_X \phi)X, \ \phi Z). \end{aligned}$$

Then from (2.2), (2.3) and the fact that $\xi \in TN_{\perp}$, we obtain

$$g(h(X, \phi X), \phi Z) = g(\bar{\nabla}_X X, \ Z) = -g(\bar{\nabla}_X Z, \ X).$$

Thus by (2.4) and (3.1), we get

$$g(h(X, \phi X), \phi Z) = -(Z \ln f)\|X\|^2. \tag{3.5}$$

Interchanging X by ϕX in (3.5) and using the fact that ξ is tangent to $N_{\perp}$, we get

$$g(h(X, \phi X), \phi Z) = (Z \ln f)\|X\|^2. \tag{3.6}$$

Thus (3.5) and (3.6) imply

$$(Z \ln f)\|X\|^2 = 0, \forall Z \in TN_{\perp} \ \& \ X \in TN_T. \tag{3.7}$$

Thus, from (3.2), (3.4) and (3.7) we conclude that f is constant i.e., M is CR-product. This completes the proof. □

Now, the other case i.e., $N_T \times {}_f N_\perp$ with ξ tangential to N_T is dealt with the following. To prove the main theorem first we obtain some useful formulae for later use.

Lemma 3.1. *Let $M = N_T \times {}_f N_\perp$ be a warped product CR-submanifold of a Lorentzian Sasakian manifold $\bar{M}$ such that ξ is tangent to N_T, where N_T and $N_\perp$ are invariant and anti-invariant submanifolds of $\bar{M}$, respectively. Then*

(i) $\xi \ln f = 0$,
(ii) $g(h(X,Y), FZ) = 0$,
(iii) $g(h(X,Z), FW) = g(h(X,W), FZ)$,
(iv) $g(h(\phi X, Z), FW) = (X \ln f) g(Z,\ W) = g(h(\phi X, W), FZ)$

for any $X, Y \in TN_T$ and $Z,\ W \in TN_\perp$.

Proof. The first part is obtained from (1.1), (2.3) and (2.4). Now for any $X \in TN_T$ and $Z \in TN_\perp$, we have

$$\nabla_X Z = \nabla_Z X = (X \ln f) Z. \tag{3.8}$$

On the other hand for any $X, Y \in TN_T$ and $Z \in TN_\perp$, by formula (2.4) we have

$$g(h(X,Y),\ \phi Z) = g(\bar{\nabla}_X Y,\ \phi Z).$$

On using (2.3) and (2.9), we get

$$\begin{aligned} g(h(X,Y),\ \phi Z) &= -g(\bar{\nabla}_X \phi Y,\ Z) = g(\phi Y,\ \bar{\nabla}_X Z) \\ &= g(\phi Y,\ \nabla_X Z). \end{aligned}$$

Taking account of the formula (3.8), the above equation yields

$$g(h(X,Y),\ \phi Z) = (X \ln f) g(\phi Y,\ Z) = 0.$$

That proves $g(h(X,Y),\ FZ) = 0$. For (iii), for any $X \in TN_T$ and $Z, W \in TN_\perp$ we have

$$\begin{aligned} g(h(X,Z),\ \phi W) &= g(\bar{\nabla}_X Z, \phi W) \\ &= -g(\bar{\nabla}_X \phi Z, W) \\ &= g(A_{\phi Z} X, W) \\ &= g(h(X,W),\ \phi Z), \end{aligned}$$

or equivalently, $g(h(X,Z),\ FW) = g(h(X,W), FZ)$. This proves (iii). Now, for any $X \in TN_T$ and $Z, W \in TN_\perp$ and using (2.2), (2.3), (2.4), (2.9) and the fact that ξ is tangent to N_T, formula (3.8) gives

$$\begin{aligned} g(\nabla_X Z, W) = g(\nabla_Z X, W) &= g(\bar{\nabla}_Z X, W) \\ &= g(\phi \bar{\nabla}_Z X, \phi W) - \eta(\bar{\nabla}_Z X)\eta(W). \end{aligned}$$

That is

$$(X \ln f)g(Z,W) = g(\bar{\nabla}_Z \phi X, \phi W) - g((\bar{\nabla}_Z \phi)X, \phi W)$$
$$= g(\nabla_Z \phi X + h(Z, \phi X), \phi W).$$

The above equation becomes

$$(X \ln f)g(Z,W) = g(h(Z,\phi X), \phi W) + (\phi X \ln f)g(Z, FW)$$
$$= g(h(Z, \phi X), \phi W).$$

This means that $(X \ln f)g(Z,W) = g(h(Z,\phi X), FW)$. This proves the first equality of (iv). For the second equality, by Gauss formula we may write

$$\begin{aligned} g(h(\phi X, Z), \phi W) &= g(\bar{\nabla}_{\phi X} Z, \phi W) \\ &= -g(\phi \bar{\nabla}_{\phi X} Z, W) \\ &= g((\bar{\nabla}_{\phi X}\phi)Z, W) - g(\bar{\nabla}_{\phi X}\phi Z, W) \\ &= g(A_{\phi Z}\phi X, W) \\ &= g(h(\phi X, W), \phi Z), \end{aligned}$$

i.e., $g(h(\phi X, Z),\ FW) = g(h(\phi X, W),\ FZ)$. This proves the lemma completely. ☐

Theorem 3.2. *Let M be a proper CR-submanifold of a Lorentzian Sasakian manifold $\bar{M}$ with integrable distribution $\mathcal{D}^{\perp}$. Then M is locally a CR-warped product if and only if*

$$A_{\phi Z}X = -(\phi X \mu)Z \tag{3.9}$$

for each $X \in \mathcal{D} \oplus \langle \xi \rangle$, $Z \in \mathcal{D}^{\perp}$ and μ, a C^{∞}-function on M such that $V\mu = 0$, for each $W \in \mathcal{D}^{\perp}$.

Proof. If M is CR-warped product submanifold $N_T \times_f N_{\perp}$, then on applying Lemma 3.1, we obtain (3.9). In this case $\mu = \ln f$.

Conversely, suppose M is a proper CR-submanifold of a Lorentzian Sasakian manifold $\bar{M}$ satisfying (3.9), then for any $X,\ Y \in \mathcal{D} \oplus \langle \xi \rangle$

$$g(h(X,Y), \phi Z) = g(A_{\phi Z}X, Y) = g(-(\phi X \mu)Z, Y) = 0$$
$$\Rightarrow \quad g(\bar{\nabla}_X \phi Y, Z) = 0,$$

which implies

$$g(\nabla_X Y, Z) = 0.$$

This means $\mathcal{D} \oplus \langle \xi \rangle$ is integrable and its leaves are totally geodesic in M. So far as anti-invariant distribution $\mathcal{D}^{\perp}$ is concerned, it is involutive on M

(cf. [1]). Moreover, for any $X \in \mathcal{D} \oplus \langle\xi\rangle$ and $Z, W \in \mathcal{D}^{\perp}$, we have

$$\begin{aligned} g(\nabla_Z W, X) &= g(\bar{\nabla}_Z W,\ X) \\ &= g(\phi\bar{\nabla}_Z W,\ \phi X) - \eta(\bar{\nabla}_Z W)\eta(X) \\ &= g(\bar{\nabla}_Z \phi W,\ \phi X) - g((\bar{\nabla}_Z \phi)W,\ \phi X) \\ &= -g(A_{\phi W} Z, \phi X) - g((\bar{\nabla}_Z \phi)W,\ \phi X). \end{aligned}$$

The second term in the right hand side of the above equation vanishes in view (2.3) and the fact that ξ tangential to N_T and the first term will be

$$-g(A_{\phi W} Z, \phi X) = -g(h(Z, \phi X), \phi W) = -g(A_{\phi W} \phi X, Z).$$

Making use of (2.1), (3.9) and Lemma 3.1 (i), the above equation takes the form

$$g(\nabla_Z W, X) = -g(A_{\phi W} Z, \phi X) = X\mu\ g(Z, W). \tag{3.10}$$

Now, by Gauss formula

$$g(h'(Z, W), X) = g(\nabla_Z W, X)$$

where h' denotes the second fundamental form of the immersion of $N_{\perp}$ into M. On using (3.10), the last equation gives

$$g(h'(Z, W), X) = X\mu\ g(Z, W).$$

The above relation shows that the leaves of $\mathcal{D}^{\perp}$ are totally umbilical in M. Moreover, the fact that $V\mu = 0$, for each $V \in \mathcal{D}^{\perp}$, implies that the mean curvature vector on $N_{\perp}$ is parallel along $N_{\perp}$ i.e., each leaf of $\mathcal{D}^{\perp}$ is an extrinsic sphere in M. Hence by virtue of a result in [9] we obtain that M is locally a CR-warped product submanifold $N_T \times\ _{\mu}N_{\perp}$ of $\bar{M}$. This proves the theorem completely. □

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References

[1] P. Alegre, *CR-submanifolds of Lorentzian manifolds*, Int. Elect. J. Geom. **2**(2009), 95-103.

[2] A. Bejancu, *Geometry of CR-submanifolds*, Kluwer, Dordrecht, (1986).

[3] R.L. Bishop and B. O'Neill, *Manifolds of Negative curvature*, Trans. Amer. Math. Soc. **145**(1969), 1-49.

[4] B.Y. Chen, *Geometry of warped product CR-submanifolds in Kaehler manifold*, Monatsh. Math. **133**(2001), 177-195.

[5] B.Y. Chen, *Geometry of warped product CR-submanifolds in Kaehler manifolds II*, Monatsh. Math. **134**(2001), 103-119.

[6] U.C. De, A. Al-Aqeel and A.A. Shaikh, *Submanifolds of Lorentzian para-Sasakian manifold*, Bull. Malays. Math. Sci. Soc. **28**(2005), 223-227.

[7] H. Gill and K.K. Dube, *Generalized CR-submanifolds of a trans Lorentzian para Sasakian manifold*, Proc. Nat. Acad. Sci. India Sect. A, **76(A)II** (2006), 119-124.

[8] I. Hasegawa and I. Mihai, *Contact CR-warped product submanifolds in Sasakian manifolds*, Geom. Dedicata **102**(2003), 143-150.

[9] S. Hiepko, *Eine Inner Kennzeichungder Produkte*, Math.Ann., **241**(1979), 209-215.

[10] V.A. Khan, K.A. Khan and Sirajuddin, *A note on warped product submanifolds of Kenmotsu manifolds*, Results Math. (Submitted).

[11] K. Matsumoto, *On Lorentzian paracontact manifolds*, Bull. Yamagata Univ. Natur. Sci. **12**(1989), 151-156.

[12] I. Mihai and R. Rosca *On Lorentzian P-Sasakian manifolds*, Classical analysis (Kazimierz Donly, 1991), 155-169, World Sci. Publ., River Edge, NJ, 1992.

[13] K. Yano and M. Kon, *Structures on manifolds*, Series in Pure Mathematics, World Scientific Publishing Co., Singapore, 1984.

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